DYNAMIC MODEL DESCRIBING THE EFFECTS OF VACCINATION WITH SATURATED INCIDENT RATE ON MEASLES EPIDEMIC

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ABSTRACT

Based on the characteristics of measles transmission, Susceptible, Infected, Recovery (SIR) epidemic model was examined to ascertain the force of saturation term and also to investigate the effects of vaccination on the measles epidemic. The model exhibits two equilibria: diseases free and endemic equilibrium points of the model were found, stability analysis and numerical simulation were carried out.

Keywords: SIR, Compartmental model, stability, equilibrium points and Vaccination.
INTRODUCTION
Measles is a highly contagious disease with more than 90% attack rate. In Africa region, World Health Organization WHO (2008) agreed on how to eliminate Measles by 2012, and this aim to improve measles surveillance system and the goals include “reducing the incidence of measles to < 5 cases/106 population per year in all countries”, “increasing the first dose of measles containing vaccine (MCV1) to greater than 90% at the national level and greater than 80% in all districts” and “measles surveillance system performance that reports non-measles febrile rash illness rate of ≥2 cases per 100,000 population per year”. According to report by World Health Organization (WHO) during 2000–2016, measles vaccination prevented an estimated 20.4 million deaths (WHO, 2017).

In Nigeria, measles case-based surveillance started in response to measles catch-up supplementary immunization activities (SIAs) in the Northern part in the last quarter of 2005 and later implemented across the country in late 2006. 1,346 suspected measles cases were reported since January 2007 with 196 laboratory-confirmed by the laboratory and or epidemiological linkage. 62% of the confirmed cases were 1-4 years and 23% aged 5-14 years. Hence the pattern of measles in Nigeria is predominantly among the younger un-immunized population due to immunity gaps as a result of inadequate routine measles coverage among others (Saleh, 2016).

Measles is vaccine preventable hence eradication efforts are geared towards vaccination of susceptible. Measles vaccination was introduced since 1963, initially as Measles - Mumps – Rubella (MMR) vaccine, it is recommended children receive the first dose of MMR1 at the age of 12 months and the second dose MMR2 is given at the age of 4 – 5 years generally through the school based vaccination programme. But in some developing countries Nigeria inclusive, with the introduction of vaccine it was thought that measles will be a disease of the past (Agumadu, 2005).

Mathematical modeling has become an essential tool in studying the spread and control of communicable diseases (Anderson and May, 1991; Zhang et al., 2013; Keeling and Rohani, 2007; Diekmann and Heesterbeek, 2000; Bashiru et al., 2014, 2017). These research works identified the disease burden and gave recommendation for improvement on surveillance and control. Mathematical models
based on the underlying transmission mechanism of measles might help the medical and scientific community understand better how the disease spreads in the community. Therefore, by developing such models, we can evaluate the potential effectiveness of different approaches to bring the epidemic under control.

In this study, we propose a mathematical model with saturated incidence rate as a system of ordinary differential equation to investigate the impact of vaccination on transmission of measles epidemic in the presence of saturated incident rate.

\[ \frac{dS}{dt} = \lambda k - \beta S I \frac{SI}{1 + mI} - \mu S \\
\frac{dI}{dt} = \beta S I \frac{SI}{1 + mI} - (\alpha + \mu) I \\
\frac{dR}{dt} = \gamma I - \mu R \]

Figure 1: Proposed Schematic diagram for the measles transmission.

where \( S(t), I(t) \) and \( R(t) \) represent the number of Susceptible, Infected and Recovered individuals at time \( t \) respectively. \( \lambda k \) is the recruitment rate of the population, \( \mu \) is the natural death rate of the population, \( \alpha \) is the disease induced death, \( \rho \) is the vaccination fraction, \( m \) is the saturated term and \( \gamma \) is the recovery rate of the infective individuals.

MATERIALS AND METHOD

The Basic Mathematical Model.

In this model, the SIR model has been adopted with the introduction of saturated incidence rate. It is assumed that individuals are equally likely to be infected by the infectious individuals, except those who are immune, everybody in our population is susceptible to the measles disease, Recovered individuals are permanently immune and the population is homogeneously mixed.
\[
\begin{align*}
\frac{dS(t)}{dt} &= \lambda k - \frac{\beta S(t)I(t)}{1 + mI(t)} - \left(\mu + \rho\right)S(t) \\
\frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1 + mI(t)} - \left(\mu + \alpha + \gamma\right)I(t) \\
\frac{dR(t)}{dt} &= \lambda(t) - \mu R(t) + \rho S(t)
\end{align*}
\]

**Derivation of \( R_0 \) Using the Next Generation Matrix**

Let \( G \) be a next generation matrix. It comprises of two parts \( F \) and \( V^{-1} \) where

\[
F = \left[ \frac{\partial F_i(x_0)}{\partial x_j} \right]
\]

\( \ldots 2 \)

\[
V = \left[ \frac{\partial V_i(x_0)}{\partial x_j} \right]
\]

\( \ldots 3 \)

\( F_i \) is the new infections, while the \( V_i \) transfers of infections from one compartment to another. \( X_0 \) is the disease free equilibrium state.

\( R_0 \) is the dominant Eigen value of the matrix

\[
G = FV^{-1}
\]

\( \ldots 4 \)

To calculate the next generation matrix for the SIR model in eq. (1), we need to enumerate the number of ways;

(1) New infections can arise

(2) Individuals can move between compartments

Though there are two disease states but only one way to create new infections. Hence, we are concerned with I and R compartment of the model. Thus

\[
\frac{dI}{dt} = \frac{\beta SI}{1 + mI} - \left(\mu + \alpha + \gamma\right)I
\]

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\[
\frac{dR}{dt} = \gamma I - \mu R + \rho S
\]

---6

From which we obtain

\[
F = \begin{pmatrix}
\frac{\beta \lambda k}{(\mu + \rho)} - (\mu + \alpha + \gamma) & 0 \\
0 & 0
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
\mu + \alpha + \gamma & 0 \\
\gamma & \mu
\end{pmatrix}
\]

---7

\[
V^{-1} = \begin{pmatrix}
\frac{1}{\mu} & 0 \\
\frac{\gamma}{\mu(\mu + \alpha + \gamma)} & \frac{1}{(\mu + \alpha + \gamma)}
\end{pmatrix}
\]

---8

\[
G = FV^{-1} = \begin{pmatrix}
\frac{1}{\mu} \left( \frac{\beta \lambda k}{(\mu + \rho)} - (\mu + \alpha + \lambda) \right) & 0 \\
0 & 0
\end{pmatrix}
\]

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Clearly, It is easy to see that the dominant eigenvalue gives the \( R_0 \)

\[
\therefore R_0 = \frac{\beta \lambda k}{(\mu + \rho)(\mu + \alpha + \gamma)}
\]

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**Steady state and local stability of the critical points.**

The local stability of the disease – free – equilibrium and endemic equilibrium of the system Eq. (1) will be discussed in this section. The system (1) has a disease free of
\[ E_0 = \left( \frac{\lambda k}{\mu + \rho}, 0, \frac{\lambda k \rho}{\mu (\mu + \rho)} \right) \]

Also, the system accepts a unique endemic equilibrium \( E_*(S_*, I_*, R_*) \)

**Proposition 1:** if \( R_0 < 1 \), then the disease free equilibrium \( E_0 \) is locally asymptotically stable.

\[ |A - I\lambda| = \begin{bmatrix} -(\mu + \rho) - \lambda & -\beta S & 0 \\ 0 & (\beta S - \mu - \alpha - \gamma) - \lambda & 0 \\ \rho & \gamma & -(\mu + \lambda) \end{bmatrix} = 0 \]

…12

Solving (12) , the eigenvalue are,
\[-(\mu + \rho + \lambda)(\beta S - \mu - \alpha - \gamma - \lambda)(-\mu - \lambda) = 0\]

…13

Clearly (13) has three roots.
\[ \lambda_1 = -(\mu + \rho), \quad \lambda_2 = -\mu \quad \text{and} \quad \lambda_3 = \beta S - \mu - \alpha - \gamma \]

Since \( S_0 = \frac{\lambda k}{\mu + \rho} \). Then \( R_0 = \frac{\beta \lambda k}{(\mu + \rho)(\mu + \alpha + \gamma)} < 1 \)

…14

Hence, the \( R_0 < 1 \), then the disease free equilibrium point \( E_0 \) is locally asymptotically stable.

**Stability Analysis of the Endemic Equilibrium.**

Let the equilibrium \( E_* = (S_*, I_*, R_*) \) where each component corresponds to an earlier value.

Let \( X = S - S_* \), \( Y = I - I_* \), \( Z = R - R_* \)

Then,
\[
\begin{align*}
\frac{dX}{dt} &= -\frac{\beta(X + S_*)(y + I_*)}{1 + mI_*} - (\mu + \rho)(X + S_*) \\
\frac{dy}{dt} &= \frac{\beta(X + S_*)(y + I_*)}{1 + mI_*} - (y + I_*)(\mu + S_*) - \gamma(y + I_*) \\
\frac{dZ}{dt} &= \gamma(y + I_*) - \mu(Z + R_*) + \rho(X + S_*)
\end{align*}
\]

Proposition 2: The unique endemic equilibrium \( E_* \) is locally asymptotically stable if \( 1 < R_0 \) and unstable if \( R_0 > 1 \).

Proof. The jacobian matrix at \( E_* \) of model (6) is

\[
|A - I\lambda| = \begin{pmatrix}
\frac{-\beta I_*}{1 + mI_*} - (\mu + \rho) - \lambda & \frac{-\beta S_*}{1 + mI_*} & 0 \\
\frac{\beta I_*}{1 + mI_*} & \frac{\beta I_*}{1 + mI_*} - (\eta + \alpha + \gamma) - \lambda & 0 \\
\rho & \gamma & -(\mu + \lambda)
\end{pmatrix}
\]

The characteristic equation of \( |A - I\lambda| \) is

\[
-\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3
\]

Where,

\[
a_1 = \frac{\beta \lambda k}{1 + mI_*} R_0 + \frac{\mu + \rho}{\mu + \alpha + \gamma} + 1 > 0
\]

\[
a_2 = R_0 \frac{\beta \lambda k}{1 + mI_*} - (\mu + \rho) \left[ \frac{1}{(\mu + \alpha + \gamma) - R_0} - \left( \mu + \frac{\beta \lambda k}{(1 + mI_*)} \right) \right] > 0
\]

\[
a_3 = (R_0 - 1) \left( \frac{\beta \lambda k}{1 + mI_*} + \mu + \rho \right) \mu > 0
\]

Considering Routh–Hurwitz criteria, it is obvious that \( a_1 a_2 - a_3 > 0 \).

Hence, the endemic equilibrium \( E_* \) is asymptotically stable.
Numerical simulation.

In order to analyze the behavior of the Eq.(1), we varied the key parameters to investigate the impact of the saturated term on the transmission dynamic of measles. We solved the system by Runge Kutta Felhberg 45 (RKF45) methods using the parameters; $$\lambda = 0.5, \beta = 0.6, \mu = 0.19, \gamma = 0.6, \alpha = 0.3, k = 30, and \ m = 0.6$$
with different values for the vaccination rate.

RESULTS AND DISCUSSION

In Figure 2, the vaccination is at rate $$\rho = 0.3$$, in fig. 3 $$\rho = 0.6$$ and in fig. 4. Other parameters were constant, in the first simulation $$\lambda = 0.5, \beta = 0.6, \mu = 0.19, \gamma = 0.6, \alpha = 0.3, k = 30, and \ m = 0.6$$
with the vaccination rate $$\rho = 0.3$$.

It can be seen from Figure 2 that if the vaccination rate $$\rho = 0.3$$ the susceptible class decreases significantly, the infected class also decreases steadily while the recovered class increases. As the vaccination increases from $$\rho = 0.3 \ \text{to} \ \ 0.6$$ , a little change occurs in all the three classes.

When the vaccination was increased further from 0.6 to 0.8 , we can see in Figure 4 that the number of susceptible individuals and infected decreased further to the equilibrium level compared to Figure 2 and Figure 3 while recovered increased to a particular level and later slightly coming down then to equilibrium level.

It can be observed from Figure 2 to Figure 4 that as the vaccination rates increased, the measles infected population and susceptible population decreased to lower level. The Figure 2 to Figure 4 above depict positive impact of vaccination on measles transmission.
Figure 2: Graph of Susceptible, Infected and Recovery when $m = 0.6$ and $\rho = 0.3$

Figure 3: Graph of Susceptible, Infected and Recovery when $m = 0.6$ and $\rho = 0.6$
CONCLUSION

This study investigated an SIR model with saturated incidence rate for the transmission of measles with vaccination. The stability behavior of the model was studied and basic reproductive number $R_0$ was defined. It was observed that the model was locally asymptotically stable when $R_0 < 1$ and measles is endemic when $R_0 > 1$. Results from this study showed that increasing the use of measles vaccine had a significant impact on the rate of measles transmission and its related complications. Increasing the measles vaccination coverage rate will further decrease the prevalence of measles and decreasing the vaccination coverage will increase the rate of transmission of measles which will affect the development of human resources in the country. Therefore, the study suggests higher vaccination rate to eradicate the disease in the society.

Figure 4: Graph of Susceptible, Infected and Recovery when $m = 0.6$ and $\rho = 0.8$
REFERENCES


Appendix.

Runge Kutta Felhberg 45 (RKF45) method using Maple software.

>`with(DEtools) :
>`with(linalg) :
>`with(plots) :

```maple
> dSdt := k - \frac{(beta + gamma \cdot T)}{1 + m \cdot H} \cdot S - (mu + rho) \cdot S;
> dHdt := \frac{(beta + gamma \cdot T)}{1 + m \cdot H} \cdot S - (delta + mu) \cdot H - gamma \cdot H;
> dRdt := gamma \cdot H - mu \cdot R + rho \cdot S;
>
eq:={diff(S(t),t)=lambda*k-(beta*S(t)*Q(t))/(1+m*Q(t))-(mu+v)*S(t),diff(Q(t),t)=-(gamma*Q(t)+(beta*S(t)*Q(t))/(1+m*Q(t))-sigma*Q(t)-mu*Q(t),diff(R(t),t)=gamma*Q(t)-mu*R(t)+v*S(t),S(0)=100,Q(0)=10,R(0)=0};
> A1:=eval(eq,{m=0.4,lambda=0.5,beta=0.6,mu=0.19,gamma=0.6,sigma=0.5,k=30,v=0.3,N=110});
> dsol:=dsolve(A1,type=numeric,stiff=true,output=array([0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20]));
> eq1:={0 = -1.29*Q(t)+.6*S(t)*Q(t)/(1+.4*Q(t)), 0 = .6*Q(t)-.19*R(t)+.3*S(t),0 = 15.0-.6*S(t)*Q(t)/(1+.4*Q(t))-49*S(t)};
> solve(eq1,{Q(t),R(t),S(t)});
> eq2:=dsolve([ Q(0) = 10, R(0) = 0, S(0) = 100, diff(Q(t), t) = -1.29*Q(t)+.6*S(t)*Q(t)/(1+.4*Q(t)), diff(R(t), t) = .6*Q(t)-.19*R(t)+.3*S(t), diff(S(t), t) = 15.0-.6*S(t)*Q(t)/(1+.4*Q(t))-49*S(t)],numeric);
```


> plots[odeplot](eq2,[[t,S(t),color=blue,linestyle=DASHDOT],[t,Q(t),color=BLACK,linestyle=LONGDASH],[t,R(t),color=RED,linestyle=LONGGDASH]],0..20,axes=NORMAL,thickness=2);